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An Elementary Proof of Error Estimates for the Composite Trapezoidal Rule over Rectangle $[a, b] \times [c, d]$

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ABSTRACT: In Numerical analysis we study the some numerical methods to find the approximate solution of problems instead than exact solution. numerical integration to find the numerical value of a definite integral. In this paper we derived trapezoidal rule over rectangular region an its error term to solve double integral.

KEYWORDS: Trapezoidal Rule, Newton divided Difference for Bivariate data, Double integral, Error.

I. INTRODUCTION

The object of this paper is to extend the trapezoidal rule formula over interval to rectangle $[a, b] \times [c, d]$and to find its error term. The error analysis of bivariate function over rectangle is derived recursively from the corresponding univariate error analysis in [3] .from such extended trapezoidal formula for bivariate function to solve double integral over given rectangular region provided that function$z = f(x, y)$must be continuous. Now trapezoidal rule over$[x_i, x_{i+1}] \times [y_j, y_{j+1}]$for uneven space

$\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(x, y)\, dxdy =$
$\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{4}\left[f(x_i, y_j) + f(x_{i+1}, y_j) +\right.$
fxi,yj+1+fxi+1,yj+1 [1]

In this paper we used Bivariate Newton divided difference interpolating polynomial for f on the set $[a, b] \times [c, d]$ such that$[a, b] = \{x_0, x_1 \dots x_m$ and $[c, d] = \{y_0, y_1, \dots, y_n]$

$P(x, y) =$
$\sum_{j=0}^{n} \sum_{i=0}^{m} f[x_0, x_1, \dots x_i; y_0, y_1, \dots, y_j] \prod_{i=0}^{m}(x -$
xi)j=0n(y−yj)[2]

and its error formula for f is continuous there exist values $\zeta \in (a, b), \eta \in (c, d)$and $(\zeta', \eta') \in (a, b) \times (c, d)$such that

$E(x, y) =$
$\frac{\prod_{i=0}^{m}(x-x_i)}{(m+1)!} \frac{\partial^{m+1} f(\zeta, y)}{\partial x^{m+1}} + \frac{\prod_{j=0}^{n}(y-y_j)}{(n+1)!} \frac{\partial^{n+1} f(x, \eta)}{\partial y^{n+1}} -$
$\frac{\prod_{i=0}^{m}(x-x_i) \prod_{j=0}^{n}(y-y_j)}{(m+1)!(n+1)!} \frac{\partial^{m+n+2} f(\zeta', y')}{\partial x^{m+1} \partial y^{n+1}}$[2]

II. MAIN RESULT

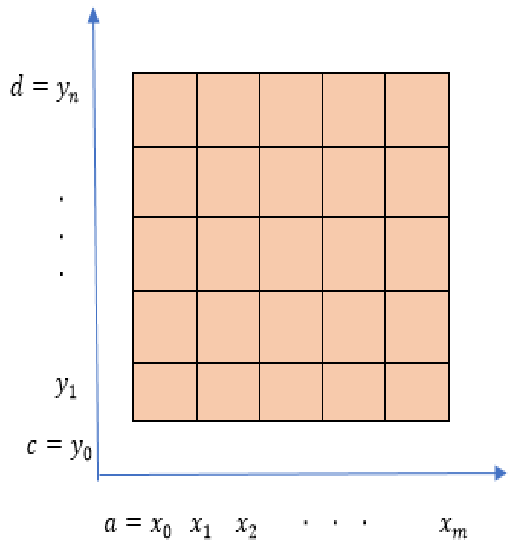
Theorem:

Let$f: [a, b] \times [c, d] \to \mathbb{R}$. Suppose that the rectangle $[a, b] \times [c, d]$is divided into mn sub-rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$of width $h = \frac{b-a}{M}$, $k = \frac{c-d}{N}$using equally spaced nodes $x_i = x_0 + ih$, $y_j = y_0 + jk$for $i = 0,1,2, \dots, m$and $j = 0,1,2, \dots, n$. The composite trapezoidal rule for mn sub rectangles is

$$T(f, hk) = \Bigg[f(x_0, y_0) + f(x_0, y_n) + f(x_m, y_0) + f(x_m, y_n) + 2 \sum_{0<i<m} \big(f(x_i, y_0) + f(x_i, y_n)\big) + 2 \sum_{0<j<n} \big(f(x_0, y_j) + f(x_m, y_j)\big) + 4 \sum_{0<j<n} \sum_{0<i<m} f(x_i, y_j)\Bigg]$$

Proof:

Let $a = x_0 < x_1 < \cdots < x_m = b$ and $c = y_0 < y_1 < \cdots < y_n = d$ be partition of $[a, b]$ and $[c, d]$



$$\int\int f(x,y)dxdy \approx \frac{hk}{4}\left(\sum_{i=0}^{m}(f(x_i,y_0)+f(x_i,y_1)) + \sum_{i=0}^{m}(f(x_i,y_1)+f(x_i,y_2)) + \cdots + \sum_{i=0}^{m}(f(x_i,y_{n-1})+f(x_i,y_n))\right)$$

$$= \frac{hk}{4}\Big[f(x_0,y_0)+f(x_m,y_0) + f(x_0,y_n)+f(x_m,y_n) + 2\big(f(x_0,y_1)+f(x_0,y_2)+\cdots + f(x_0,y_{n-1})\big) + 2\big(f(x_m,y_1)+f(x_m,y_2)+\cdots + f(x_m,y_{n-1})\big) + 2\big(f(x_1,y_0)+f(x_2,y_0)+\cdots + f(x_{m-1},y_0)\big) + 2\big(f(x_1,y_n)+f(x_2,y_n)+\cdots + f(x_{n-1},y_{n-1})\big) + 4\big(f(x_1,y_1)+f(x_2,y_1)+\cdots + f(x_{m-1},y_1)+\cdots.+f(x_1,y_{n-1}) + f(x_2,y_{n-1})+\cdots + f(x_{m-1},y_{n-1})\big)\Big]$$

$$T(f,hk) = \frac{hk}{4}\Bigg[f(x_0,y_0)+f(x_0,y_n)+f(x_m,y_0) + f(x_m,y_n) + 2\sum_{0<i<m}\big(f(x_i,y_0)+f(x_i,y_n)\big) + 2\sum_{0<j<n}\big(f(x_0,y_j)+f(x_m,y_j)\big) + 4\sum_{0<j<n}\sum_{0<i<m} f(x_i,y_j)\Bigg]$$

Corollary:(Trapezoidal Rule:Error Analysis)

Suppose that rectangle$[a,b]\times[c,d]$is subdivide intomn sub-rectangles $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ of width $h=\frac{b-a}{M}$, $k=\frac{c-d}{N}$. Let M and N sub intervals of $[a,b]$and $[c,d]$ resp.we divide the intervals into same number of intervals to form a rectangle i.e $M = N$ .The composite trapezoidal rule

$$T(f,hk) = \frac{hk}{4}\Bigg[f(x_0,y_0)+f(x_0,y_n)+f(x_m,y_0) + f(x_m,y_n) + 2\sum_{0<i<m}\big(f(x_i,y_0)+f(x_i,y_n)\big) + 2\sum_{0<j<n}\big(f(x_0,y_j)+f(x_m,y_j)\big) + 4\sum_{0<j<n}\sum_{0<i<m} f(x_i,y_j)\Bigg]$$

is an approximation to the integral

$$\int_c^d\int_a^b f(x,y)dxdy = T(f,hk)+E_T(f,hk)$$

Furthermore, if $f\in C^{(1,1)}([a,b]\times[c,d])$,there exist $\zeta\in(a,b), \eta\in(c,d)$ and $(\zeta',y')\in(a,b)\times(c,d)$ so that error term $E_T(f,hk)$ has the form

$E_T(f,hk) = -\left(\frac{1}{12N^2}\frac{\partial^2 f(\zeta,y)}{\partial x^2} + \frac{1}{12N^2}\frac{\partial^2 f(x,\eta)}{\partial y^2} + \right.$
1144N3∂4f(ζ′,y′)∂x2∂y2

Proof:

We determine the error term when the rule is applied over $[x_0,x_1]\times[y_0,y_1]$. Integrating the Langrange polynomial for bivariate data $P_{11}(x,y)$ and its remainder yields

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)dxdy = \int_{y_0}^{y_1}\int_{x_1}^{x_1} P_{11}(x,y) + \int_{y_0}^{y_1}\int_{x_1}^{y_0} E(x,y)$$

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)dxdy$$
$$= \int_{y_0}^{y_1}\int_{x_0}^{x_1}\big(f[x_0;y_0] + f[x_0;y_0,y_1](y-y_0)$$
$$+ f[x_0,x_1;y_0](x-x_0)$$
$$+ f[x_0,x_1;y_0,y_1](x-x_0)(y-y_0)\big)dxdy$$
$$+ \int_{y_0}^{y_1}\int_{x_1}^{y_0}\Bigg(\frac{\prod_{i=0}^{1}(x-x_i)}{2!}\frac{\partial^2 f(\zeta,y)}{\partial x^2}$$
$$+ \frac{\prod_{j=0}^{1}(y-y_j)}{2!}\frac{\partial^2 f(x,\eta)}{\partial y^2}$$
$$- \frac{\prod_{i=0}^{1}(x-x_i)\prod_{j=0}^{1}(y-y_j)}{2!\,2!}\frac{\partial^4 f(\zeta',y')}{\partial x^2\partial y^2}\Bigg)dxdy$$
$$= \frac{hk}{4}[f(x_0,y_0) + f(x_0,y_1) + f(x_1,y_0)$$
$$+ f(x_1,y_1)] + \frac{(y_1-y_0)}{2}\left(\frac{-(x_1-x_0)^3}{6}\right)\frac{\partial^2 f(\zeta,y)}{\partial x^2}$$
$$+ \frac{(x_1-x_0)}{2}\left(\frac{-(y_1-y_0)^3}{6}\right)\frac{\partial^2 f(x,\eta)}{\partial y^2}$$
$$- \frac{1}{4}\left(\frac{-(x_1-x_0)^3}{6}\right)\left(\frac{-(y_1-y_0)^3}{6}\right)\frac{\partial^4 f(\zeta',y')}{\partial x^2\partial y^2}$$

$$= \frac{hk}{4}[f(x_0,y_0) + f(x_0,y_1) + f(x_1,y_0)$$
$$+ f(x_1,y_1)]$$
$$- \Bigg(\frac{(y_1-y_0)(x_1-x_0)^3}{12}\frac{\partial^2 f(\zeta,y)}{\partial x^2}$$
$$+ \frac{(x_1-x_0)(y_1-y_0)^3}{12}\frac{\partial^2 f(x,\eta)}{\partial y^2}$$
$$+ \frac{(x_1-x_0)^3(y_1-y_0)^3}{144}\frac{\partial^4 f(\zeta',y')}{\partial x^2\partial y^2}\Bigg)$$

Now we add the error terms for all sub-rectangles $[x_{i-1},x_i]\times[y_{j-1},y_j]$, sowe obtain

$$\int_c^d\int_a^b f(x,y) = \sum_{j=0}^{n}\sum_{i=0}^{m}\left(\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} f(x,y)dxdy\right)$$
$$= \sum_{j=0}^{n}\sum_{i=0}^{m}\frac{(x_i-x_{i-1})(y_j-y_{j-1})}{4}\big(f(x_{i-1},y_{j-1})$$
$$+ f(x_i,y_{j-1}) + f(x_{i-1},y_j) + f(x_i,y_j)\big)$$
$$- \Bigg(\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{(y_j-y_{j-1})(x_i-x_{i-1})^3}{12}\frac{\partial^2 f(\zeta_i,y)}{\partial x^2}$$
$$+ \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{(x_i-x_{i-1})(y_j-y_{j-1})^3}{12}\frac{\partial^2 f(x,\eta_j)}{\partial y^2}$$
$$+ \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{(x_i-x_{i-1})^3(y_j-y_{j-1})^3}{144}\frac{\partial^4 f(\zeta'_i,y'_j)}{\partial x^2\partial y^2}\Bigg)$$

M and N number of intervals of $[a,b]$ and $[c,d]$ resp. we divide the interval into same number of intervals that is $M = N$ and points are uniformly spaced with $(x_i - x_{i-1}) = \frac{1}{N}and(y_j - y_{j-1}) = \frac{1}{N}$

$$= \sum_{j=0}^{n}\sum_{i=0}^{m}\frac{(x_i-x_{i-1})(y_j-y_{j-1})}{4}\big(f(x_{i-1},y_{j-1})$$
$$+ f(x_i,y_{j-1}) + f(x_{i-1},y_j)$$
$$+ f(x_i,y_j)\big)$$
$$- \Bigg(\frac{N}{12N^3}\left[\frac{1}{N}\sum_{i=0}^{m}\frac{\partial^2 f(\zeta_i,y)}{\partial x^2}\right]$$
$$+ \frac{N}{12N^3}\left[\frac{1}{N}\sum_{i=0}^{m}\frac{\partial^2 f(x,\eta_j)}{\partial y^2}\right]$$
$$+ \frac{N}{144N^4}\left[\frac{1}{N^2}\sum_{i=0}^{m}\frac{\partial^4 f(\zeta'_i,y'_j)}{\partial x^2\partial y^2}\right]\Bigg)$$

The terms in square bracket can be recognized as an average of values for the second order andmixed order partial derivatives and hence it's replaced by $\frac{\partial^2 f(\zeta,y)}{\partial x^2}, \frac{\partial^2 f(x,\eta)}{\partial y^2}$ and $\frac{\partial^4 f(\zeta',y')}{\partial x^2\partial y^2}$

$$= \sum_{j=0}^{n}\sum_{i=0}^{m}\frac{(x_i-x_{i-1})(y_j-y_{j-1})}{4}\big(f(x_{i-1},y_{j-1})$$
$$+ f(x_i,y_{j-1}) + f(x_{i-1},y_j)$$
$$+ f(x_i,y_j)\big)$$
$$- \Bigg(\frac{1}{12N^2}\frac{\partial^2 f(\zeta,y)}{\partial x^2}$$
$$+ \frac{1}{12N^2}\frac{\partial^2 f(x,\eta)}{\partial x^2}$$
$$+ \frac{1}{144N^3}\frac{\partial^2 f(\zeta',\eta')}{\partial x^2}\Bigg)$$

Assume that the second order and mixed order partial derivatives $\frac{\partial^2 f(\zeta,y)}{\partial x^2}$, $\frac{\partial^2 f(x,\eta)}{\partial y^2}$ and $\frac{\partial^4 f(\zeta',y')}{\partial x^2\partial y^2}$ are bounded by $K_1, K_2$and K_3 respectively. i.e$\left|\frac{\partial^2 f(x,\eta)}{\partial y^2}\right| \le K_1$, $\left|\frac{\partial^2 f(\zeta,y)}{\partial x^2}\right| \le K_2$, $\left|\frac{\partial^4 f(\zeta',y')}{\partial x^2\partial y^2}\right| \le K_3$

$$\left|\int_c^d\int_a^b f(x,y)dxdy - T(f,hk)\right|$$
$$= \Bigg|-\Bigg(\frac{1}{12N^2}\frac{\partial^2 f(\zeta,y)}{\partial x^2}$$
$$+ \frac{1}{12N^2}\frac{\partial^2 f(x,\eta)}{\partial y^2}$$
$$+ \frac{1}{144N^3}\frac{\partial^4 f(\zeta',y')}{\partial x^2\partial y^2}\Bigg)\Bigg|$$

$$|E_T(f,hk)| \le \frac{1}{12N^2}\left|\frac{\partial^2 f(\zeta,y)}{\partial x^2}\right| + \frac{1}{12N^2}\left|\frac{\partial^2 f(x,\eta)}{\partial y^2}\right| + \frac{1}{144N^3}\left|\frac{\partial^4 f(\zeta',y')}{\partial x^2 \partial y^2}\right|$$

$$|E_T(f,hk)| \le \frac{K_1}{12N^2} + \frac{K_2}{12N^2} + \frac{K_3}{144N^3}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{m} \frac{(x_i - x_{i-1})(y_j - y_{j-1})}{4}\Big(f(x_{i-1},\ y_{j-1}) + f(x_i, y_{j-1}) + f(x_{i-1}, y_j) + f(x_i, y_j)\Big) - \left(\frac{1}{12N^2}\frac{\partial^2 f(\zeta,y)}{\partial x^2} + \frac{1}{12N^2}\frac{\partial^2 f(x,\eta)}{\partial y^2} + \frac{1}{144N^3}\frac{\partial^4 f(\zeta',y')}{\partial x^2 \partial y^2}\right)$$

III. APPLICATIONS:

In this section we solve some problems and its error term by this newly developed formula.

1. Consider $\int\int xy^3 dxdy$ **over** $[0,2]\times[0,1]$

Solution: Exact value of above integral is $\frac{2}{3} \approx 0.666666666667$

Following table shows that the number of subinterval along x-axis and y-axis is doubled and subinterval size is reduced by factor $\frac{1}{2}$. Such that successive errors $E_T(f,hk)$ are diminished by approximately $\frac{1}{4}$

Number of rectangles	**h,k**	$T(f,hk)$	$E_T(f,hk) = \|I(f) - T(f,hk)\|$	$\frac{E_{2h2k}}{E_{hk}}$
1	2.0, 1.0	1.0	0.3333333333333333	
4	1.0, 0.5	0.75	0.08333333333333326	4.000000000000004
16	0.5, 0.25	0.6875	0.02083333333333326	4.0000000000000043
64	0.25, 0.125	0.671875	0.005208333333333259	4.00000000000043
256	0.125 , 0.0625	0.66796875	0.001302083333333259	4.00000000000171
1024	0.0625, 0.03125	0.6669921875	$3.255208333332593 10^{-4}$	4.000000000000681
4096	0.03125, 0.015625	0.666748046875	$8.138020833325932 10^{-5}$	4.000000000002728
16384	0.015625, 0.0078125	0.66668701171875	$2.034505208325932 10^{-5}$	4.000000000010914
65536	0.0078125, 0.00390625	0.6666717529296875	$5.086263020759318 10^{-6}$	4.000000000043657

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262144	0.00390625, 0.001953125	0.6666679382 324219	$1.271565755134318 10^{-6}$	4.000000000174624
1048576	0.001953125, 0.0009765625	0.6666669845 581055	$3.178914387280685 10^{-7}$	4.00000000069849

2. Consider $\int\int x^3y^3 + xy$over $[1,2]\times[2,3]$
Solution:
Exact value of above integral is $\frac{1035}{16} = 64.6875$

Number of rectangles	h,k	$T(f,hk)$	$E_{hk} = \lvert I(f) - T(f,hk)\rvert$
1	1.0, 1.0	82.5	17.8125
4	0.5, 0.5	68.96484375	4.27734375
16	0.25, 0.25	65.745849609375	1.058349609375
64	0.125, 0.125	64.95140075683594	0.2639007568359375
256	0.0625, 0.0625	64.75343227386475	0.0659322738647461
1024	0.03125, 0.03125	64.70398038625717	0.01648038625717163
4096	0.015625, 0.015625	64.69161992892623	0.004119928926229477
16384	0.0078125, 0.0078125	64.6885299717542	0.0010299717542068265
65536	0.00390625, 0.00390625	64.68775749228355	0.00025749228355520965
262144	0.001953125, 0.001953125	64.68756437303026	$6.437303025563779\times10^{-5}$
1048576	0.0009765625, 0.0009765625	64.6875160932529	$1.609325289564367\times10^{-5}$

$\frac{\partial^2 f(x,y)}{\partial x^2} = 6xy^3 + y$, $\frac{\partial^2 f(x,y)}{\partial y^2} = 6x^3y$, $\frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2} = 36xy$

$max_{1\leq\zeta\leq2}\left|\frac{\partial^2 f(\zeta,y)}{\partial x^2}\right| = \frac{\partial^2 f(2,3)}{\partial x^2} = 327 = K_1,$

$max_{2\leq\eta\leq3}\left|\frac{\partial^2 f(x,\eta)}{\partial y^2}\right| = \frac{\partial^2 f(2,3)}{\partial y^2} = 144 = K_2$,

$$\max_{\substack{1\leq\zeta\leq2\\2\leq\eta\leq3}}\left|\frac{\partial^4 f(\zeta,\eta)}{\partial x^2\partial y^2}\right| = \frac{\partial^2 f(2,3)}{\partial x^2\partial y^2} = 216 = K_3$$

For number of subintervals of interval $[1,2]$ and $[2,3]$ is 1024. i.e $N = M = 1024$

$$|E_T(f,hk)| \leq \frac{K_1}{12N^2} + \frac{K_2}{12N^2} + \frac{K_3}{144N^3} = \frac{327}{12N^2} + \frac{144}{12N^2} + \frac{216}{144N^3} = 3.74331139028072 4\times10^{-5}$$

IV. CONCLUSION

The numerical error results obtained by this formula and by regular calculations are closely related.

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